Einstein-Yang-Mills Solitons: The Role of Gravity

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Abstract

The canonical Bartnik-McKinnon solitons are regular solutions of the coupled Einstein-Yang-Mills system in which gravity may balance the repulsive nature of the Yang-Mills field. We examine the role played by gravity in balancing the system and determine its strength. In particular, we obtain an analytic lower bound on the fundamental mass-to-radius ratio, $\max_r\{2m(r)/r\} > 2/3$, which is a necessary condition for the existence of globally regular Einstein-Yang-Mills solitons. Our analytical results are in accord with numerical calculations.

The wide interest in the theory of gravitating solitons and hairy black holes started after the (numerical) discovery of globally regular particle-like solutions of the coupled Einstein-Yang-Mills (EYM) system [1, 2]. This was rather unexpected, since it is well known that when taken apart, neither the vacuum Einstein equations nor the Yang-Mills (YM) equations have nontrivial static globally regular solutions. For the pure YM theory in flat space, this was proved in [3, 4]. The corresponding result for vacuum Einstein gravity is Lichnerowicz's theorem [5]. Indeed, the unique static spherically symmetric solution of pure Einstein gravity is the celebrated Schwarzschild metric, which is singular at the origin.

The EYM solitons (also known as Bartnik-McKinnon solitons [1]) can be thought of as equilibrium states of a pair of physical fields [6], one of which is repulsive (the YM field) while the other one is attractive (gravity). It turns out that the YM repulsive force can balance gravitational attraction and prevent the formation of singularities in spacetime. From a mathematical point of view, it is the nonlinearity of the YM equations which may preclude spacetime singularities.

The fact that only the coupled EYM system can have regular solitons (while the pure YM equations can have no static regular solutions) raises the fundamental question: how strong must gravity be in order to balance the repulsive nature of the YM field? Here we shall give an analytic answer to this question by proving the existence of a lower bound on the fundamental mass-to-radius ratio, $\max_r\{2m(r)/r\}$. This quantity is central for determining the spacetime geometry of the solution and the strength of the gravitational interaction.

The line element of a static spherically symmetric spacetime may take the following form in Schwarzschild coordinates [7]

$$ds^2 = -e^{-2\delta}\mu dt^2 + \mu^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{1}$$

where the metric functions δ and $\mu = 1 - 2m(r)/r$ depend only on the Schwarzschild radius r. Here $m(r)$ is the mass contained within a sphere of radius r. (We use gravitational units in which $G = c = 1$.) Asymptotic flatness requires that as $r \to \infty$,

$$\mu(r) \to 1 \quad and \quad \delta(r) \to 0 \ , \tag{2}$$

and a regular center requires

$$\mu(r) = 1 + O(r^2) \ . \tag{3}$$

The Einstein equations, $G^\mu_\nu = 8\pi T^\mu_\nu$, reads

$$\mu' = 8\pi r T^t_t + (1-\mu)/r \ , \tag{4}$$

$$\delta' = 4\pi r(T^t_t - T^r_r)/\mu \ , \tag{5}$$

where the prime stands for differentiation with respect to r. The conservation equation, $T^\mu_{\nu;\mu} = 0$, has only one nontrivial component [7]

$$T^\mu_{r;\mu} = 0 \ . \tag{6}$$

Substituting Eqs. (4) and (5) in Eq. (6), one obtains

$$(e^{-\delta} r^4 T^r_r)' = \frac{e^{-\delta} r^3}{2\mu}[(3\mu - 1)(T^r_r - T^t_t) + 2\mu T] \ , \tag{7}$$

where T is the trace of the energy momentum tensor. Below we shall concentrate on the function $\mathcal{E}(r) \equiv e^{-\delta} r^4 T^r_r$.

Our main focus here is on the canonical EYM solitons. However, our analytical results would also be valid for any Einstein-matter theory in which the matter fields satisfy the following energy conditions:

- The weak energy condition (WEC). This means that the energy density, $\rho \equiv -T^t_t$, is positive semidefinite and that it bounds the pressures, in particular, $|T^r_r| \leq \rho$. This implies the inequality $T^r_r - T^t_t \geq 0$.

- The energy density ρ goes to zero faster than r^{-4}. This requirement is the natural way to impose the condition that there are no extra conserved charges (besides the ADM mass) defined at asymptotic infinity associated with the matter fields [7]. (We recall that the charges defined at spatial infinity, like the electric charge of the Reissner-Nordström solution in Einstein-Maxwell theory, are associated with the $\rho \sim r^{-4}$ asymptotic behavior.)

- The trace of the energy momentum tensor is nonnegative, $T \geq 0$.

We point out that the EYM system satisfies these energy conditions, in particular, $\rho(r \to \infty) \sim r^{-6}$ and $T = 0$.

Taking cognizance of Eq. (7) together with the boundary conditions, Eqs. (2)-(3), and the above energy conditions one finds that as $r \to 0^+$,

$$\mathcal{E} \to 0^+ \quad and \quad \mathcal{E}' \geq 0 \ , \tag{8}$$

and that as $r \to \infty$,

$$\mathcal{E} \to 0^- \quad and \quad \mathcal{E}' \geq 0 \ , \tag{9}$$

Equations (8) and (9) imply that there is a finite interval, $r_0 \leq r \leq r_1$, in which $\mathcal{E}'(r) < 0$. (The function $\mathcal{E}$ switches signs from positive values to negative values within this interval). For $\mathcal{E}'(r)$ to be negative, we must have $3\mu(r) - 1 < 0$ in this interval. This follows from the WEC and the assumption that $T \geq 0$. This, in turn, yields a lower bound on the maximal mass-to-radius ratio of the regular soliton:

$$\text{max}_r \Big\{ \frac{2m(r)}{r} \Big\} > \frac{2}{3} \ . \tag{10}$$

The dimensionless quantity $2m(r)/r$ is fundamental for determining the spacetime geometry and the strength of the gravitational interaction. The analytically derived inequality therefore sets a lower bound on the strength of gravity which is required in order to stabilize the repulsive nature of the YM field.

We would like to point out that *upper* bounds on the ratio $\text{max}_r\{2m(r)/r\}$ are well known in the literature, the famous of all is the Buchdahl inequality, $\text{max}_r\{2m(r)/r\} > 8/9$ [8, 9]. However, this upper bound has been derived for matter fields which satisfy the conditions $T \leq 0$ and $T^r_r \geq 0$, and it should be emphasized that this assumption is violated by the YM system considered here (we have proved that T^r_r must switch signs for the EYM system, and this analytical prediction is in accord with numerical calculations, see [6].) To our best knowledge, the inequality (10) is the first *lower* bound derived on the fundamental ratio $\text{max}_r\{2m(r)/r\}$.

Let us confirm the validity of the analytical relation, Eq. (10), with the help of available numerical data. It turns out [10] that $\text{max}_r\{2m/r\} = 0.76 > 2/3$ for the canonical $n = 1$ EYM soliton (the positive integer n is the number of nodes of the matter fields.) In addition,

the function $\max_r\{2m/r\}$ increases as n increases. These numerical results are obviously in accord with our analytical bound, Eq. (10).

Summary.— The nonlinearity of the YM equations may preclude spacetime singularities, and thus allows the existence of globally regular solitons solutions of the coupled EYM system. On the other hand, this nonlinearity has restricted most former studies of the EYM equations to the numerical regime. It is therefore highly important to obtain some *analytical* insights about this nonlinear system. This was the main purpose of our analysis.

It is well known that the pure YM equations have no regular static solutions due to the repulsive nature of the YM field. It is the attractive nature of gravity that allows the existence of regular EYM solitons. One would therefore like to examine the role played by gravity in balancing the coupled system. Here we have proved analytically that the fundamental mass-to-radius ratio must be bounded according to $\max_r\{2m(r)/r\} > 2/3$ in order to balance the system and to allow the existence of globally regular solitons. We find it remarkable that such a simple and transparent relation emerged out of the highly nonlinear equations.

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